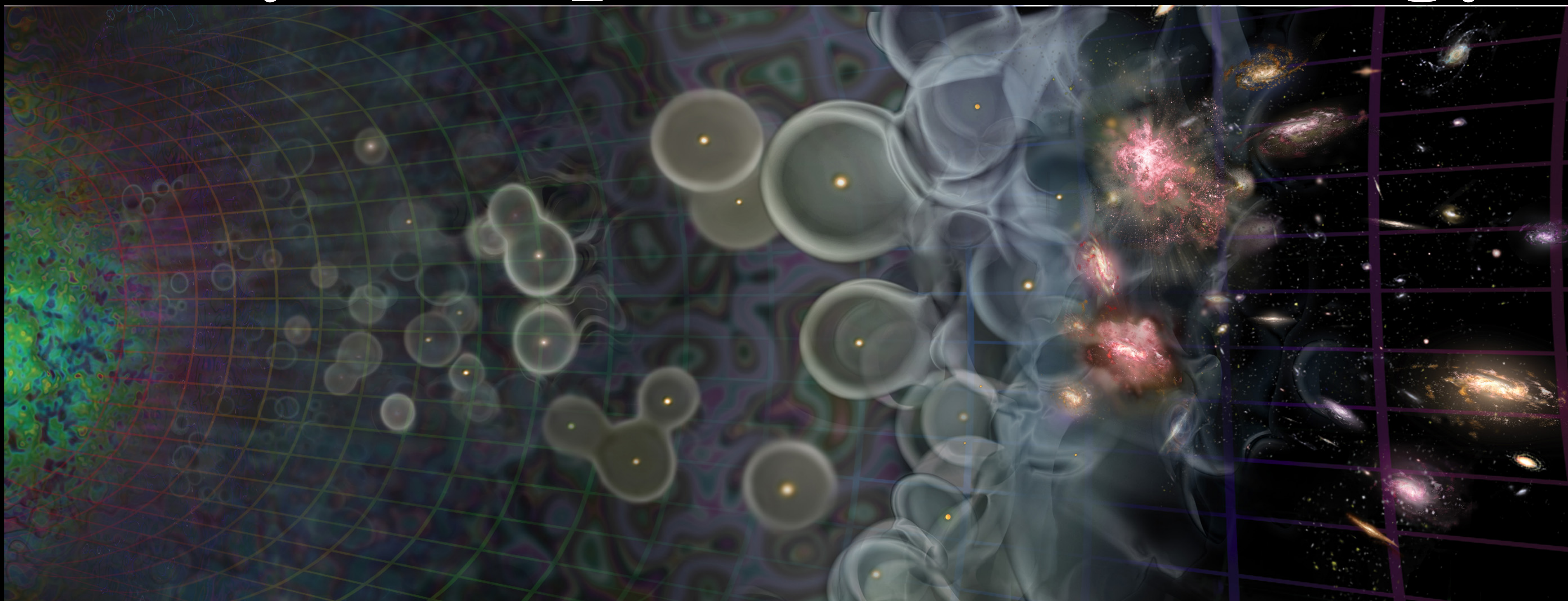
Hamiltonian diagonalization in hybrid quantum cosmology



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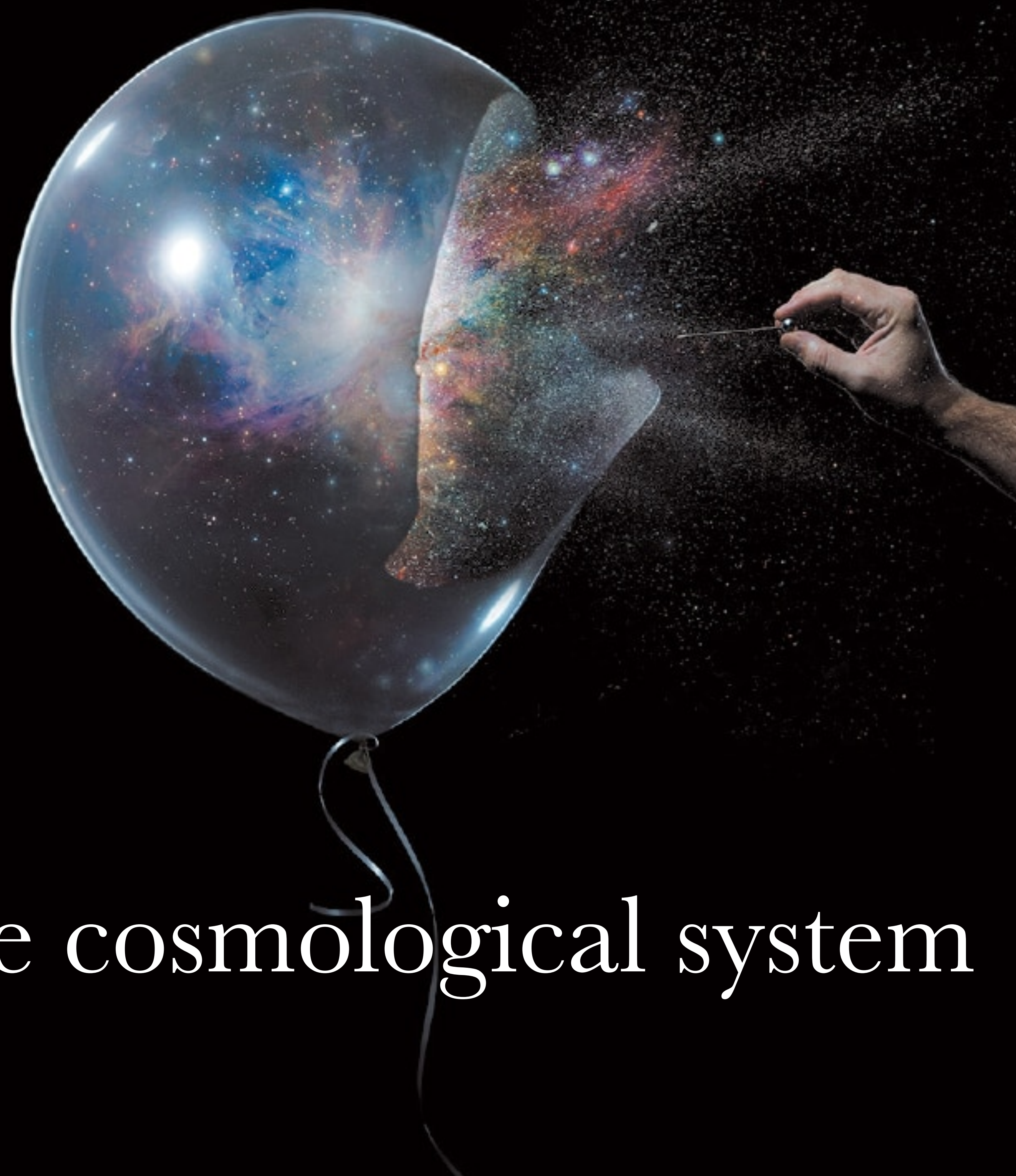
Motivation

- Traditional QFT suffers from UV issues.
- Can be traced to the assumption of classical spacetime backgrounds: could quantum gravity change the picture?
- Hybrid scheme for cosmological models:
    - ★ QG representation for homogeneous d.o.f..
    - ★ Fock representation for inhomogeneities.

Motivation

- Traditional QFT suffers from UV issues.
- Could quantum gravity change the picture?
- Hybrid scheme for cosmological models.
- Freedom in performing canonical transformation that assign different dynamical roles to the homogeneous sector and the inhomogeneities.
- Well-defined Fock contrib. to Hamiltonian: Restrictions on the choices of annihilation and creationlike variables.
- Further constraints on the dynamical characterization of the vacua for gauge-invariant perturbations: Diagonalization of their contribution to the Hamiltonian constraint.



The cosmological system

The cosmology

- FLRW spacetime with flat & compact hypersurfaces.
- Minimally coupled homogeneous scalar field (inflaton).
- Scalar and tensor perturbations of the metric & inflaton.
- Expansion in spatial Laplace eigenmodes $(l_0\vec{k}/(2\pi) \in \mathbb{Z}^3)$.
- Canonical formalism: truncation of the action at quadratic order in <u>all</u> the perturbations.
- Hamiltonian: linear combination of constraints:

  Zero-mode of Hamiltonian constr. & linear perturbative constr.

The cosmology

- Background-dependent canonical transformations lead to:
  - ⋆ Mukhanov-Sasaki scalar gauge-invariants.
  - ⋆ Abelianized perturbative constraints (& momenta).

- Can be completed to be canonical for the entire system by:

$$(a, \pi_a) \longrightarrow (\breve{a}, \pi_{\breve{a}}) = (a - \Delta a, \pi_a - \Delta\pi_a)$$
$$(\phi, \pi_\phi) \longrightarrow (\breve{\phi}, \pi_{\breve{\phi}}) = (\phi - \Delta\phi, \pi_\phi - \Delta\pi_\phi)$$

quadratic in perturbations

- Express the entire Hamiltonian in terms of the new canonical set, keeping the quadratic truncation.

Hamiltonian

- Hamiltonian: new linear perturbative constraints and zero-mode of the Hamiltonian constraint, which is:

$$[H_{|0} + {}^{s}H_{|2} + {}^{T}H_{|2}](\breve{a}, \pi_{\breve{a}}, \breve{\phi}, \pi_{\breve{\phi}})$$

$$H_{|0} = \frac{1}{2 l_0^3 a^3}\left[\pi_\phi^2 - \frac{4\pi G}{3} a^2 \pi_a^2 + 2 l_0^6 a^6 V(\phi)\right] \longrightarrow \text{FLRW constraint}$$

$${}^{s}H_{|2} = \frac{1}{2a}\sum_{\vec{k}\neq 0}\left[\left(k^2 + s^{(s)}\right)|v_{\vec{k}}|^2 + |\pi_{v_{\vec{k}}}|^2\right] \longrightarrow \text{MS Hamiltonian}$$

$${}^{T}H_{|2} = \frac{1}{2a}\sum_{\vec{k}\neq 0,\epsilon}\left[\left(k^2 + s^{(t)}\right)|d_{\vec{k},\epsilon}|^2 + |\pi_{d_{\vec{k},\epsilon}}|^2\right] \longrightarrow \text{Tensor Hamiltonian}$$

- The terms $s^{(s)}$ and $s^{(t)}$ are the background-dependent "masses" for the Mukhanov-Sasaki and tensor perturb.

Hamiltonian: issues

- Annihilation and creationlike variables for MS perturb.:

$$a_{\vec{k}} = f_k v_{\vec{k}} + g_k \bar{\pi}_{v_{\vec{k}}}, \qquad \bar{a}_{\vec{k}} = \bar{f}_k \bar{v}_{\vec{k}} + \bar{g}_k \pi_{v_{\vec{k}}}$$

- They satisfy the correct Poisson algebra iff

$$f_k \bar{g}_k - g_k \bar{f}_k = -i$$

- In terms of them, the MS Hamiltonian takes the form:

$${}^{s}H_{|2} = \frac{1}{2a} \sum_{\vec{k}\neq 0} \left\{ 2\left[\left(k^2 + s^{(s)}\right)|g_k|^2 + |f_k|^2 \right] \bar{a}_{\vec{k}} a_{\vec{k}} - \left[\left(k^2 + s^{(s)}\right) g_k^2 + f_k^2 \right] \bar{a}_{\vec{k}} \bar{a}_{-\vec{k}} + \text{H.c.} \right\}$$

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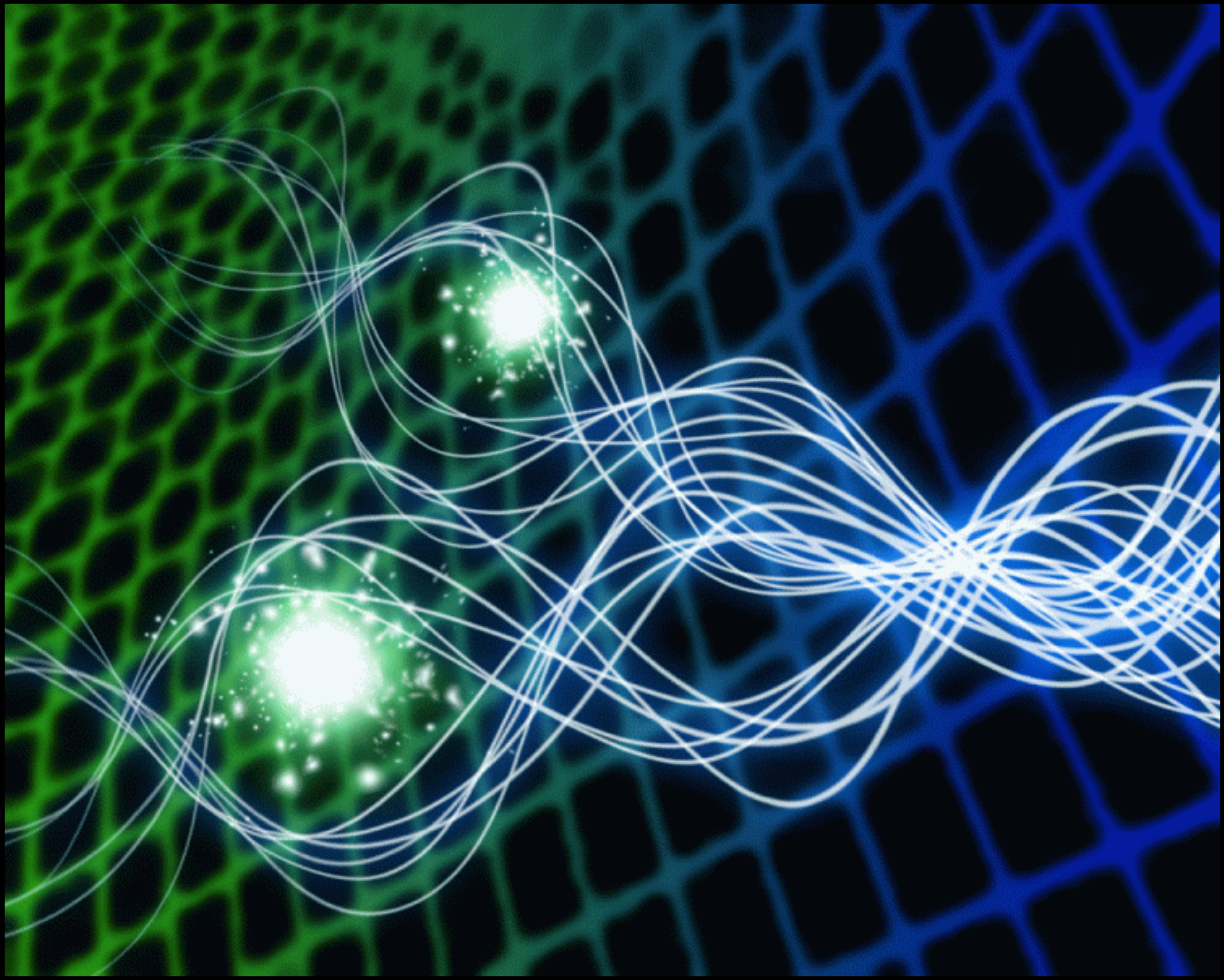
- Cannot happen due to background dependence of $s^{(s)}$.

New variables: diagonalization



New gauge-invariants

- Considering the system as a whole, freedom in:
  - ⋆ Dynamical separation of homogeneous geometry and gauge-invariants via canonical transformations.
  - ⋆ Choice of Fock vacuum for the fermionic perturbations, within the hybrid scheme.

- This ambiguity can be encoded in choices of the form:

$$a_{\vec{k}} = f_k(\breve{a}, \pi_{\breve{a}}, \breve{\phi}, \pi_{\breve{\phi}}) v_{\vec{k}} + g_k(\breve{a}, \pi_{\breve{a}}, \breve{\phi}, \pi_{\breve{\phi}}) \bar{\pi}_{v_{\vec{k}}}$$

that can be completed again into a canonical set for the entire cosmology by introducing

$$(\breve{a}, \pi_{\breve{a}}) \longrightarrow (\tilde{a}, \pi_{\tilde{a}}), \qquad (\breve{\phi}, \pi_{\breve{\phi}}) \longrightarrow (\tilde{\phi}, \pi_{\tilde{\phi}}),$$

correcting again the new homogeneous variables with certain contributions quadratic in perturbations.

New Hamiltonian

- The resulting (new) Mukhanov-Sasaki Hamiltonian ${}^s\tilde{H}_{|2}$ is the old one ${}^sH_{|2}$ plus the contribution:

$$\frac{1}{2}\sum_{\vec{k}\neq 0}\left[2\mathrm{Re}\left(\bar{f}_k\{g_k, H_{|0}\} - \bar{g}_k\{f_k, H_{|0}\}\right)\bar{a}_{\vec{k}}a_{\vec{k}} + \left(g_k\{f_k, H_{|0}\} - f_k\{g_k, H_{|0}\}\right)\bar{a}_{\vec{k}}\bar{a}_{-\vec{k}} + \mathrm{H.c.}\right]$$

and its Fock representation can be well-defined on vacuum.

- In fact, the interaction terms in the Hamiltonian for each Fourier scale k are completely eliminated iff:

$$k^2 + s^{(s)} + h_k^2 - a\{h_k, H_{|0}\} = 0, \qquad h_k = g_k^{-1} f_k$$

which is a semilinear PDE whose complex solutions satisfy:

$$\mathrm{Im}(h_k)^2 = k^2 + s^{(s)} - \frac{\mathrm{Im}(h_k)''}{2\mathrm{Im}(h_k)} + \frac{3}{4}\left[\frac{\mathrm{Im}(h_k)'}{\mathrm{Im}(h_k)}\right]^2, \quad {}' = a\{., H_{|0}\}$$

Hamiltonian diagonalization

$$k^2 + s^{(s)} + h_k^2 - a\{h_k, H_{|0}\} = 0, \qquad h_k = g_k^{-1} f_k$$

- Given a solution, the norm of f_k is fixed by the requirement of annihilation and creation-like algebra as:

$$2|f_k|^2 = -|h_k|^2[\mathrm{Im}(h_k)]^{-1}$$

- All remaining freedom is thus codified in the phase F_k of f_k.

- This phase can be constrained by imposing nice properties of the resulting diagonal Hamiltonian, which is:

$${}^s\tilde{H}_{|2} = \frac{1}{a}\sum_{\vec{k}\neq 0} \Omega_k \bar{a}_{\vec{k}} a_{\vec{k}}, \qquad \Omega_k = a\{H_{|0}, F_k\} - (k^2 + s^{(s)})\frac{\mathrm{Im}(h_k)}{|h_k|^2}$$

e.g., its positivity as a function of the background d.o.f..

Asymptotic diagonalization

$$k^2 + s^{(s)} + h_k^2 - a\{h_k, H_{|0}\} = 0, \qquad h_k = g_k^{-1} f_k$$

- An asymptotic analysis of the Hamiltonian reveals that diagonalization is obtained, when $k \to \infty$, with:

$$k g_k = i f_k \left[1 - \frac{1}{2k^2} \sum_{n=0}^{\infty} \left(\frac{-i}{2k} \right)^n \gamma_n \right], \qquad \gamma_0 = s^{(s)},$$

$$\gamma_{n+1} = a\{H_{|0}, \gamma_n\} + 4 s^{(s)} \left[\gamma_{n-1} + \sum_{l=0}^{n-3} \gamma_l \gamma_{n-(l+3)} \right] - \sum_{l=0}^{n-1} \gamma_l \gamma_{n-(l+1)}, \qquad \forall n \geq 0$$

- If the series converges, and the result can be extended to all Fourier scales, it should select a solution for the PDE.

- In the linearized context of QFT in curved spacetimes, our asymptotic characterization leads to: the Minkowski vacuum, in the case of constant mass; and the Bunch-Davies vacuum, when the homogeneous background is fixed as the de Sitter solution.

Conclusions

- Separation of the phase space in the phase space of hybrid quantum cosmology so that the Hamiltonian for gauge-invariant perturbations is diagonalized.
- Leads to a very specific dynamical characterization of annihilation and creation-like variables, in the UV regime.
- Freedom left in one phase, that can be fixed in order to obtain a quantum Hamiltonian operator with nice properties.
- For all Fourier scales, the problem is reduced to solving a semilinear first order PDE w.r.t the homogeneous d.o.f..
- In the context of QFT in curved spacetimes, the imaginary part of the solution satisfies the typical nonlinear second order equation for the frequency of normalised solutions to the gauge-invariants' equations.